



General equilibrium cryptocurrency pricing in an OLG model

Bertrand Crettez, Lisa Morhaim

► To cite this version:

Bertrand Crettez, Lisa Morhaim. General equilibrium cryptocurrency pricing in an OLG model. Mathematical Social Sciences, 2022, 117, pp.13–19. <10.1016/j.mathsocsci.2022.02.003>. <hal-04103599>

HAL Id: hal-04103599

<https://univ-panthéon-assas.hal.science/hal-04103599v1>

Submitted on 22 Jul 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY-NC 4.0 - Attribution - Non-commercial use - International License



Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/mss



General equilibrium cryptocurrency pricing in an OLG model[☆]

Bertrand Crettez^{*}, Lisa Morhaim¹

Université Panthéon-Assas, Paris II, CRED, EA 7321, 21 Rue Valette - 75005 Paris, France



ARTICLE INFO

Article history:

Received 6 May 2021

Received in revised form 6 December 2021

Accepted 24 February 2022

Available online 4 March 2022

Keywords:

Cryptocurrency

Central bank currency

General equilibrium

Overlapping generations model

Asset pricing

ABSTRACT

We study the valorization of cryptocurrency in an overlapping generations model where agents have the choice between crypto- and central bank currency. We pay attention to three features of cryptocurrency: it is a risky asset (because of frauds and hacks), which one acquires by bearing transaction costs, and its supply reaches its upper limit in a finite time (e.g., Bitcoin). We consider both the standard case with linear-quadratic utility functions and the more general case where utility functions are not specified. We find that the effect of an increase in cryptocurrency on the currency prices depends on the transaction costs that individuals bear to get cryptocurrency. In particular, when these costs are convex non-increasing in the quantity of cryptocurrency, the cryptocurrency price may increase with this quantity. Otherwise, when the transaction costs are either increasing or concave non-increasing, the cryptocurrency price decreases with the quantity of cryptocurrency. We also show that the cryptocurrency and the central bank currency prices always move in opposite directions.

© 2022 Elsevier B.V. All rights reserved.

1. Introduction

We study the valorization of a cryptocurrency in an overlapping generations model where agents have the choice between cryptocurrency and central bank currency. We pay attention to three features of cryptocurrency: it is a risky asset (because of frauds and hacks), which one acquires by bearing transaction costs, and its supply reaches its upper limit in a finite time (e.g., Bitcoin). We consider both the standard case with linear-quadratic utility functions and the more general case where utility functions are not specified. In both cases, the only source of uncertainty in the model is the hack risk. This source of uncertainty, however, is compatible with deterministic currency prices. Concentrating on these prices, we find that the effect of an increase in cryptocurrency on the currency prices depends on the transaction costs that individuals bear to get cryptocurrency. In particular, when these costs are convex non-increasing in the quantity of cryptocurrency, the cryptocurrency price may increase with this quantity. Otherwise, when the transaction costs are either increasing or concave non-increasing, the cryptocurrency price decreases with its supply. We also show that the

cryptocurrency and the central bank currency prices always move in opposite directions.

This paper contributes to a burgeoning literature devoted to the economics of cryptocurrency. Biais et al. (2018) present an overlapping generations equilibrium model of cryptocurrency pricing and confront it to data on bitcoin transactional benefits and costs. Their model emphasizes that the fundamental value of the cryptocurrency is the stream of net transactional benefits it will provide, which depend on its future prices. The link between future and present prices implies that returns can exhibit large volatility unrelated to fundamentals. Consistent with the model, estimated transactional net benefits explain a statistically significant fraction of bitcoin returns.² Also in an OLG setting, Garratt and Wallace (2018) study the indeterminacy of the equilibrium exchange rates between two currencies by introducing storage costs for the central bank currency and a risk of currency crash for the cryptocurrency. Their conclusion is that little can be said about the relative value of several cryptocurrencies. Yet another OLG model is studied in Saleh (2018). He contrasts Proof-of-work blockchains with Proof-of-burn ones.³ Saleh shows that Proof-of-burn blockchains result in less price volatility than Proof-of-work's.

Schilling and Uhlig (2019) also obtain an indeterminacy result albeit in a model with two types of agents who have infinite

[☆] This research has been conducted within the Research Chair “Digital Finance” under the aegis of the Risk Foundation, a joint initiative by Groupe des Cartes Bancaires CB, La Banque Postale, Telecom Paris and Paris II Panthéon-Assas University.

^{*} Corresponding author.

E-mail address: bertrand.crettez@u-paris2.fr (B. Crettez).

¹ We thank two anonymous referees for helpful and stimulating comments on a previous version of this work.

Table 1
Macroeconomics of cryptocurrency.

Authors	Type/Model	Nb of agents	Specific features
Biais et al. (2018)	OLG	3	Theory/empirical
Fernández-Villaverde and Sanches (2019)	Search	3	Multiple cryptocurrencies
Garratt and Wallace (2018)	OLG	1	Risk of currency crash
Schilling and Uhlig (2019)	Infinite horizon	2	Random endowments
Chiu and Koepl (2017)	Infinite horizon	3	History of transactions
Saleh (2018)	OLG	1/2	Proof-of-burn blockchain

decision-horizon (they also obtain conditions under which there is no speculation in the cryptocurrencies).⁴ In a search setting, Fernández-Villaverde and Sanches (2019) extend the model by Lagos and Wright (2005) to multiple currencies and study the stability and welfare implications of the private supply of money alongside a government-backed currency.⁵ Fernandez-Villaverde and Sanches show that the existence of an upper bound on the available supply of each brand makes privately issued money in the form of cryptocurrencies consistent with price stability in a competitive environment. Table 1 compares the main approaches used in the literature.

This paper builds on Biais et al. (2018) but differs from it by studying the effects of a change in the quantity of cryptocurrency on the equilibrium currency prices and by assuming that cryptocurrency reaches its upper-limit in finite time. We also advance the study of the effect of a hack risk on the currency prices.⁶ While the magnitude of the hacking risk may be thought of as second order relatively to the large current volatility of cryptocurrency prices, we believe that this risk is likely to be a major concern in the event that cryptocurrencies are used on a large scale.

The remaining part of the paper unfolds as follows. Section 2 lays out our version of an OLG model with cryptocurrency and a risk of hacking. Section 3 is devoted to the study of the equilibrium. More precisely, because the policy setting is stable (the currency growth rates are constant) we believe that it is relevant to study equilibria where price expectations are stable too. This is why we first concentrate on the steady-state equilibrium that occurs when the cryptocurrency supply reaches its upper-limit. We illustrate this equilibrium with an example in Section 4. Section 5 briefly studies the interim dynamics between the date at which the cryptocurrency is introduced and the date at which its supply reaches its upper-limit. The interim prices are obtained by a backward induction argument from the steady state prices and the changes in the supplies of crypto- and central bank currency. Section 6 briefly concludes the paper.

2. The model

Following Biais et al. (2018), we consider an overlapping generations economy with two assets, namely cryptocurrency and central bank currency. Let X_t and m_t denote the supply of cryptocurrency and the supply of central bank currency at date t , respectively. We let μ be the constant growth rate of central bank currency.

There are three kinds of agents: the investors, the miners (who produce new coins of cryptocurrency), and the hackers. The

last two categories of agents only live one period, whereas the investors live two periods.

More specifically, at each date t , from date 0 until date T , a new generation of miners is born. They exchange new cryptocurrency coins $X_{t+1} - X_t$ (and transaction fees) against a real good which they consume at date t . As they live for only one period, they do not save. For simplicity, we assume that the miners' activity can be described by a function $\rho : X_{t+1} - X_t = \rho(X_t)$, where $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is such that $X_T = \bar{X}$, X_0 being given. Those assumptions mean that the cryptocurrency stock reaches its maximum in finite time (akin to the Bitcoin).⁷

A new generation of hackers is also born at each date t . They steal a fraction of the cryptocurrency supply X_t . The fraction of X_t which is stolen at date t is a random variable \tilde{h}_t , whose realizations h_t are in $[0, 1]$.⁸ The random variables \tilde{h}_t are i.i.d across time. Hackers sell the stolen coins against consumption goods. Like the miners they do not save.

In addition to the miners and the hackers, a unit mass of identical investors is born at any date t . Investors live two periods and their intertemporal utility function is written $U(c_t^y) + V(c_{t+1}^o)$, where $U : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the utility obtained from consumption c_t^y during youth and $V : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ the utility obtained from consumption c_{t+1}^o during old age. The utility function U satisfies $U' \geq 0$ and $U'' \leq 0$ (the same properties hold for $V(\cdot)$). We let e^y denote the investors' endowment of consumption good in youth, and e^o their endowment of consumption good in old age.⁹ Both endowments e^y and e^o are positive.

By selling a share of their endowment during youth, investors can buy q_t units of cryptocurrency at a unit price equal to p_t , as well as \hat{q}_t units of central bank currency at a unit price equal to \hat{p}_t . The consumption good is the *numéraire*. When buying cryptocurrency, investors bear a real unit cost equal to $p_t \phi(X_t)$. This cost depends on the quantity of cryptocurrency X_t and can be either decreasing or increasing, depending on the existence of economies of scales. We suppose that $\phi(0) = 0$.

The budget constraints of the investors in both youth and old age read¹⁰

$$c_t^y = e^y - p_t q_t - \hat{p}_t \hat{q}_t - \phi(X_t) p_t q_t, \quad (1)$$

$$c_{t+1}^o = e^o + (1 - h_{t+1}) p_{t+1} q_t + \hat{p}_{t+1} \hat{q}_t. \quad (2)$$

The increase in central bank currency finances government expenditures \hat{G}_t . That is, $(m_t - m_{t-1}) \hat{p}_t = \hat{G}_t$ for all t . For the sake of simplicity, those expenditures do not enter the individuals' objective functions.

⁴ In their model the real value of the central bank currency is held constant.

⁵ Chiu and Koepl (2017) use a similar search model to study the optimal design of a cryptocurrency protocol.

⁶ Biais et al. (ibid) consider thefts but not major attack on the system. Budish (2018) addresses the risk of a major attack on a cryptocurrency while Athey et al. (2016) take into account the probability that the cryptocurrency has a defect (all ongoing transactions being lost). Chiu and Koepl (2017) carefully address the problem that a blockchain must solve, namely keeping an exact history of transactions. Here, we focus on the hack risk.

⁷ For the Bitcoin, $T \sim 2022$ and $\bar{X} = 21$ millions bitcoins.

⁸ The hack risk h_t is the only source of uncertainty in the model.

⁹ Assuming that individuals' endowments are stationary considerably simplifies the study of the equilibrium.

¹⁰ Notice that from the point of view of date t , c_{t+1}^o is a random variable because of the hack risk. It will turn out, however, that the equilibrium prices p_{t+1} and \hat{p}_{t+1} are perfectly expected (see the proof of Proposition 6).

3. Equilibrium

3.1. General considerations

Let us study the problem solved by the young investor living at date t . Assume that all the prices of date $t + 1$ are perfectly expected at date t . The young investor's problem is written as follows

$$\begin{aligned} & \mathcal{P}(e_t^y, e_{t+1}^o, p_t, p_{t+1}, \tilde{h}_{t+1}) \\ &= \begin{cases} \max_{(q_t, \hat{q}_t) \in \mathbb{R}_+^2} & U(c_t^y) + \mathbb{E} \left[V(c_{t+1}^o(\tilde{h}_{t+1})) \right] \\ \text{s.t.} & c_t^y = e^y - p_t q_t - \hat{p}_t \hat{q}_t - \phi(X_t) p_t q_t \geq 0, \\ & c_{t+1}^o(\tilde{h}_{t+1}) \\ &= e^o + (1 - \tilde{h}_{t+1}) p_{t+1} q_t + \hat{p}_{t+1} \hat{q}_t \geq 0. \end{cases} \end{aligned}$$

The assumption that $(q_t, \hat{q}_t) \in \mathbb{R}_+^2$ means that young investors cannot go short in the cryptocurrency and money. This assumption is innocuous as in equilibrium only young investors hold these assets. To put it another way, a young investor cannot borrow because either her lender will no longer be alive in the next period or this lender is a young investor too, and is thus identical to her.

Now, assuming that p_t and p_{t+1} are positive and since \tilde{h}_{t+1} is a random variable taking its values in $[0, 1]$, the second inequality is always satisfied as long as $(q_t, \hat{q}_t) \in \mathbb{R}_+^2$ and therefore the young investor's problem reduces to

$$\begin{cases} \max_{(q_t, \hat{q}_t) \in \mathbb{R}_+^2} & U(e^y - p_t q_t - \hat{p}_t \hat{q}_t - \phi(X_t) p_t q_t) \\ & + \mathbb{E} \left[V \left(e^o + (1 - \tilde{h}_{t+1}) p_{t+1} q_t + \hat{p}_{t+1} \hat{q}_t \right) \right] \\ \text{s.t.} & e^y - p_t q_t - \hat{p}_t \hat{q}_t - \phi(X_t) p_t q_t \geq 0. \end{cases}$$

Proposition 1. Assume that p_t and \hat{p}_t are positive. Then the young investor's program has a unique solution.

Proof. The proof is standard. Existence comes from the fact that the objective function is continuous on the set $E = \{(q_t, \hat{q}_t) \in \mathbb{R}_+^2, p_t q_t + \hat{p}_t \hat{q}_t + \phi(X_t) p_t q_t \leq e^y\}$ and that E is a compact set (as it is a closed set included in $[0, \frac{e^y}{p_t}] \times [0, \frac{e^y}{\hat{p}_t}]$). Uniqueness comes from the objective function being strictly concave. \square

The first-order conditions for an interior solution are written as follows (for q_t and \hat{q}_t , respectively)

$$\begin{aligned} & -U' \left(e^y - p_t q_t - \hat{p}_t \hat{q}_t - \phi(X_t) p_t q_t \right) (1 + \phi(X_t)) p_t \\ & + \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1}) p_{t+1} q_t + \hat{p}_{t+1} \hat{q}_t \right) \right] \\ & \times (1 - \tilde{h}_{t+1}) p_{t+1} \Big] = 0. \end{aligned} \quad (3)$$

$$\begin{aligned} & -U' (e^y - p_t q_t - \hat{p}_t \hat{q}_t - \phi(X_t) p_t q_t) \hat{p}_t \\ & + \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1}) p_{t+1} q_t + \hat{p}_{t+1} \hat{q}_t \right) \hat{p}_{t+1} \right] = 0. \end{aligned} \quad (4)$$

Let us now consider the general equilibrium at date t . Since all investors are identical, the market clearing conditions read

$$X_t = X_t - X_{t-1} + h_t X_{t-1} + X_{t-1} - h_t X_{t-1} = q_t \quad (5)$$

$$m_t = (1 + \mu) m_{t-1} = m_{t-1} + m_t - m_{t-1} = \hat{q}_t \quad (6)$$

$$\begin{aligned} e^y + e^o &= c_t^y + c_t^y + p_t [(X_t - X_{t-1}) + X_t \phi(X_t) + h_t X_{t-1}] \\ &+ \hat{p}_t G_t. \end{aligned} \quad (7)$$

The first equation above describes the equilibrium on the market for cryptocurrency (the left-hand side represents the supply of cryptocurrency, and the right-hand side its demand). The second equation describes the equilibrium on the market for central bank currency (with the same pattern as for cryptocurrency). And the last equation describes the equilibrium on the market for the consumption good. By Walras law, when the first two markets are in equilibrium so is the third one.

Let us now focus on the equilibrium prices. Since the aggregate quantity of both currencies are strictly positive at each date, using the first-order conditions (3) and (4) we can express the equilibrium conditions at date t as

$$\begin{aligned} & -U' \left(e^y - p_t X_t - \hat{p}_t m_t - \phi(X_t) p_t X_t \right) (1 + \phi(X_t)) p_t \\ & + \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1}) p_{t+1} X_t + \hat{p}_{t+1} m_t \right) \right] \\ & \times (1 - \tilde{h}_{t+1}) p_{t+1} \Big] = 0 \end{aligned} \quad (8)$$

and

$$\begin{aligned} & -U' \left(e^y - p_t X_t - \hat{p}_t m_t - \phi(X_t) p_t X_t \right) \hat{p}_t \\ & + \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1}) p_{t+1} X_t + \hat{p}_{t+1} m_t \right) \hat{p}_{t+1} \right] = 0. \end{aligned} \quad (9)$$

Let \hat{z}_t denote the *real* value of central bank currency, i.e., $\hat{z}_t = \hat{p}_t \times m_t$. Since $m_{t+1} = (1 + \mu) m_t$, i.e., $\hat{p}_{t+1} m_t = \hat{p}_{t+1} \frac{m_{t+1}}{1 + \mu} = \frac{1}{1 + \mu} \hat{z}_{t+1}$, the previous equations can be written as

$$\begin{aligned} & -U' \left(e^y - p_t X_t - \hat{z}_t - \phi(X_t) p_t X_t \right) (1 + \phi(X_t)) p_t \\ & + \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1}) p_{t+1} X_t + \frac{\hat{z}_{t+1}}{1 + \mu} \right) \right] \\ & \times (1 - \tilde{h}_{t+1}) p_{t+1} \Big] = 0 \end{aligned} \quad (10)$$

and

$$\begin{aligned} & -U' \left(e^y - p_t X_t - \hat{z}_t - \phi(X_t) p_t X_t \right) \hat{p}_t \\ & + \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1}) p_{t+1} X_t + \frac{\hat{z}_{t+1}}{1 + \mu} \right) \hat{p}_{t+1} \right] = 0. \end{aligned} \quad (11)$$

As m_t is positive for any date t , the two foregoing equations can be rewritten as¹¹

$$\begin{aligned} & U' \left(e^y - p_t X_t - \hat{z}_t - \phi(X_t) p_t X_t \right) (1 + \phi(X_t)) p_t \\ & = p_{t+1} \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1}) p_{t+1} X_t + \frac{\hat{z}_{t+1}}{1 + \mu} \right) (1 - \tilde{h}_{t+1}) \right], \end{aligned} \quad (12)$$

$$\begin{aligned} & U' \left(e^y - p_t X_t - \hat{z}_t - \phi(X_t) p_t X_t \right) \hat{p}_t m_t \\ & = \beta \hat{p}_{t+1} m_t \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1}) p_{t+1} X_t + \frac{\hat{z}_{t+1}}{1 + \mu} \right) \right]. \end{aligned} \quad (13)$$

Now, using again the equations $z_t = \hat{p}_t m_t$, $m_{t+1} = (1 + \mu) m_t$, $\hat{p}_{t+1} m_t = \hat{p}_{t+1} \frac{m_{t+1}}{1 + \mu} = \frac{1}{1 + \mu} \hat{z}_{t+1}$, we obtain the following two equilibrium conditions

$$\begin{aligned} & U' \left(e^y - p_t X_t - \hat{z}_t - \phi(X_t) p_t X_t \right) (1 + \phi(X_t)) p_t \\ & = p_{t+1} \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1}) p_{t+1} X_t + \frac{\hat{z}_{t+1}}{1 + \mu} \right) (1 - \tilde{h}_{t+1}) \right], \end{aligned} \quad (14)$$

¹¹ Recall that all the future prices at $t + 1$ are perfectly expected.

$$U'(e^y - p_t X_t - \hat{z}_t - \phi(X_t) p_t X_t) \hat{z}_t = \frac{\hat{z}_{t+1}}{1+\mu} \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1}) p_{t+1} X_t + \frac{\hat{z}_{t+1}}{1+\mu} \right) \right]. \quad (15)$$

To contrast the equilibrium conditions above with those considered in [Garratt and Wallace \(2018\)](#) suppose that cryptocurrency and central bank currency are equal to X and m , respectively (thus $\mu = 0$). In addition, assume that $\phi(X) = 0$ and $\tilde{h}_{t+1} \equiv 0$. Then, one can see that the last two equilibrium conditions boil down to

$$U'(e^y - p_t X - \hat{p}_t m) p_t = p_{t+1} \mathbb{E} \left[V' \left(e^o + p_{t+1} X + \hat{p}_{t+1} m \right) \right], \quad (16)$$

$$U'(e^y - p_t X - \hat{p}_t m) \hat{p}_t m = \hat{p}_{t+1} m \mathbb{E} \left[V' \left(e^o + p_{t+1} X + \hat{p}_{t+1} m \right) \right]. \quad (17)$$

Further assume that $p_t = p_{t+1}$ and $\hat{p}_t = \hat{p}_{t+1}$. Then the two conditions above reduce to the ensuing one

$$U'(e^y - p_t X - \hat{p}_t m) = \mathbb{E} \left[V' \left(e^o + p_t X + \hat{p}_t m \right) \right]. \quad (18)$$

Inspecting the preceding equation, one sees that only the value of $p_t X + \hat{p}_t m$ can be determined. As shown by [Garratt and Wallace \(2018\)](#), there is indeterminacy with regard to the specific values of p and \hat{p} . That is because, the two assets are perfect substitutes. Yet, as we shall see, because we assume transaction costs for holding cryptocurrency, theft risks and different currency growth rates, both currency prices are well determinate in our setting.

3.2. Steady-state

Let us consider the date T at which the stock of cryptocurrency $(X_t)_{t \geq T}$ remains constant, that is: $X_t = \bar{X}$ for all t greater than or equal to T . While there might be equilibria where the cryptocurrency price fluctuates because of, say, sunspots and so on, we believe that where the different currency stocks evolve in a simple, predictive way, e.g., their growth rates are constant through times, the stability of the policy and institutional settings should stabilize price expectations. Therefore, we focus on the study of stationary equilibrium prices.

More formally, we look for a pair (p, \hat{z}) , where p is the value of cryptocurrency and \hat{z} the real value of central bank currency, such that for any $t \geq T$ ¹²

$$U'(e^y - p\bar{X} - \hat{z} - \phi(\bar{X})p\bar{X})(1 + \phi(\bar{X})p) = p \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1})p\bar{X} + \frac{\hat{z}}{1+\mu} \right) (1 - \tilde{h}_{t+1}) \right] \quad (19)$$

$$U'(e^y - p\bar{X} - \hat{z} - \phi(\bar{X})p\bar{X})\hat{z} = \frac{\hat{z}}{1+\mu} \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1})p\bar{X} + \frac{\hat{z}}{1+\mu} \right) \right]. \quad (20)$$

More precisely, we look for a pair of positive values of p and \hat{z} such that

$$U'(e^y - p\bar{X} - \hat{z} - \phi(\bar{X})p\bar{X})(1 + \phi(\bar{X})p) = \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1})p\bar{X} + \frac{\hat{z}}{1+\mu} \right) (1 - \tilde{h}_{t+1}) \right] \quad (21)$$

$$U'(e^y - p\bar{X} - \hat{z} - \phi(\bar{X})p\bar{X}) = \frac{1}{1+\mu} \mathbb{E} \left[V' \left(e^o + (1 - \tilde{h}_{t+1})p\bar{X} + \frac{\hat{z}}{1+\mu} \right) \right]. \quad (22)$$

The next result provides a necessary condition for the existence of a steady-state equilibrium.

Proposition 2. *If there exists $(p, \hat{z}) \in (\mathbb{R}_+^*)^2$ satisfying Eqs. (21) and (22), then*

$$1 \leq \frac{1+\mu}{1+\phi(\bar{X})} \mathbb{E} [1 - \tilde{h}_{t+1}] \text{ and } \phi(\bar{X}) \leq \mu. \quad (23)$$

Moreover,

(1) *if the utility function for old-age V is strictly concave, then both inequalities are strict,*

(2) *if $\mathbb{E} [\tilde{h}_{t+1}] > 0$, then the second inequality is strict.*

Proof. Recall that for two random variables X and Y $\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$ and for any $c, d \in \mathbb{R}$, $\text{Cov}(cX + d, Y) = c\text{Cov}(X, Y)$. For simplicity of notations, we denote by V' the random variable $V'(e^o + (1 - \tilde{h}_{t+1})\bar{X}p + \frac{\hat{z}}{1+\mu})$.

Assume that there exists $(p, \hat{z}) \in (\mathbb{R}_+^*)^2$ satisfying Eqs. (21) and (22). Then

$$\mathbb{E} [V' \cdot (1 - \tilde{h}_{t+1})] = \text{Cov}(V', 1 - \tilde{h}_{t+1}) + \mathbb{E} [V'] \mathbb{E} [(1 - \tilde{h}_{t+1})] \quad (24)$$

$$= -\text{Cov}(V', \tilde{h}_{t+1}) + \mathbb{E} [V'] \mathbb{E} [(1 - \tilde{h}_{t+1})] \quad (25)$$

$$= -\text{Cov}(V', \tilde{h}_{t+1})$$

$$+ U'(e^y - p\bar{X} - \hat{z} - p\phi(\bar{X})\bar{X})(1 + \mu)$$

$$\times \mathbb{E} [(1 - \tilde{h}_{t+1})] \text{ (by Eq. (22))}. \quad (26)$$

Using Eq. (21) we obtain

$$U'(e^y - p\bar{X} - \hat{z} - p\phi(\bar{X})\bar{X}) = \frac{1}{1+\phi(\bar{X})} \left((1 + \mu) U'(e^y - p\bar{X} - \hat{z} - p\phi(\bar{X})\bar{X}) \times \mathbb{E} [(1 - \tilde{h}_{t+1})] - \text{Cov}(V', \tilde{h}_{t+1}) \right). \quad (27)$$

Now, since when $p \in \mathbb{R}_+$ the function $\tilde{h}_{t+1} \mapsto V'(e^o + (1 - \tilde{h}_{t+1})\bar{X}p + \hat{z})$ is nondecreasing. Now, as the covariance $\text{Cov}(V', \tilde{h}_{t+1})$ is nonnegative¹³ and $U'(\cdot) > 0$, we have

$$1 \leq \frac{1+\mu}{1+\phi(\bar{X})} \mathbb{E} [1 - \tilde{h}_{t+1}]. \quad (28)$$

It follows that $\mathbb{E} [\tilde{h}_{t+1}] \leq \frac{\mu - \phi(\bar{X})}{1+\mu}$ and $\phi(\bar{X}) \leq \mu$. Moreover, condition (1) follows from the previous proof and the fact that $\tilde{h}_{t+1} \mapsto V'(e^o + (1 - \tilde{h}_{t+1})\bar{X}p + \hat{z})$ being increasing. Condition (2) is straightforward. \square

To understand [Proposition 2](#), notice that if in equilibrium the two assets are to be held by a young investor, then the marginal expected utility associated with these assets should be equal. Now, the expected return of holding one unit of currency is $1/(1 + \mu)$ since the currency grows at a constant rate μ and the price of this currency decreases by the same amount. As

¹² That is, the steady-state equilibrium is achieved at date T .

¹³ For a proof one can refer to 10.2.7 in [Schmidt \(2006\)](#) which can also be found in Theorem p.3 in [Schmidt \(2003\)](#).

for cryptocurrency, the expected return of holding one unit of cryptocurrency is $\mathbb{E}[1 - \tilde{h}_{t+1}]/(1 + \phi(\bar{X}))$. Since the hack risk is such that \tilde{h}_{t+1} is in $[0, 1]$, cryptocurrency is only held if $\mu > \phi(\bar{X})$. Otherwise, the transaction cost of cryptocurrency would be greater than the depreciation rate of the currency, and since holding crypto is risky, only holding central bank currency would be the best decision. The gist of the proposition, therefore, is that provided that the currency growth rate is greater than the transaction cost, holding crypto is a valuable choice if the hack risk is not too high.

4. A linear-quadratic example

To obtain clear-cut results as to the effect of an increase in both currency stocks we shall assume that $U(\cdot)$ is linear and $V(\cdot)$ is quadratic, that is, $V(x) = \frac{1}{2}ax^2 + bx + c$ where x is in $[0, -\frac{b}{a}]$, with $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}_+$ and $V'(e^0) = ae^0 + b > 0$.

Proposition 2 shows that $\mathbb{E}[\tilde{h}_{t+1}] \leq \frac{\mu - \phi(\bar{X})}{1 + \mu}$ is a necessary condition for the existence of positive equilibrium prices. **Proposition 3** asserts that such condition combined with an additional one is also sufficient.

Proposition 3. *When the stock of cryptocurrency is equal to \bar{X} from T on, there exists a unique pair of stationary equilibrium values $(p, \hat{z}) \in (\mathbb{R}_+^*)^2$ if, and only if, the following conditions hold*

$$(i) \ 1 - \frac{1 + \mu}{1 + \phi(\bar{X})} \mathbb{E}[(1 - \tilde{h}_{t+1})] < 0 \quad (29)$$

$$(ii) \ -(1 + \phi(\bar{X}))\mathbb{E}[1 - \tilde{h}_{t+1}] + (1 + \mu)(1 - \mathbb{E}[\tilde{h}_{t+1}])^2 - (ae^0 + b)\mathbb{V}(\tilde{h}_{t+1}) < 0. \quad (30)$$

The stationary equilibrium values of the cryptocurrency price and the central bank currency value are then as follows

$$p = \frac{1}{a\mathbb{V}[\tilde{h}_{t+1}]\bar{X}} \left(1 + \phi(\bar{X}) - (1 + \mu)\mathbb{E}[(1 - \tilde{h}_{t+1})] \right) \quad (31)$$

$$\begin{aligned} \hat{z} &= \frac{1 + \mu}{a\mathbb{V}[\tilde{h}_{t+1}]} \\ &\times \left(-(1 + \phi(\bar{X}))\mathbb{E}[1 - \tilde{h}_{t+1}] \right. \\ &\left. + (1 + \mu)(1 - \mathbb{E}[\tilde{h}_{t+1}])^2 - (ae^0 + b)\mathbb{V}(\tilde{h}_{t+1}) \right). \quad (32) \end{aligned}$$

Proof. See the [Appendix](#). \square

The first inequality above is satisfied if the necessary condition given in [Proposition 2](#) holds. The second inequality is not always true. It is only satisfied if μ is not too large, which is rather intuitive: if the money growth rate is too large, it is not worthwhile to hold central bank currency.

Note that the following conditions ensure that the equilibrium pair $(p, \hat{z}) \in (\mathbb{R}_+^*)^2$ given by (31) and (32) is feasible

$$p\bar{X} + \frac{\hat{z}}{1 + \mu} \leq -\frac{b}{a} - e^0 \quad (33)$$

$$(1 + \phi(\bar{X}))p\bar{X} + \hat{z} \leq e^y, \quad (34)$$

i.e., one needs to assume that endowment in youth e^y is large enough and the endowment in old age is small enough.

The following Proposition focuses on the properties of the equilibrium value of cryptocurrency.

Proposition 4.

(a) *The equilibrium price of cryptocurrency decreases with a mean-preserving increase in its risk (i.e., in $\mathbb{V}(\tilde{h}_{t+1})$).*

(b) *If ϕ is increasing, then the equilibrium price of cryptocurrency decreases with its stock.*

(c) *If ϕ is non-increasing, then the effect of an increase in the quantity of cryptocurrency upon its equilibrium price is indetermined, except if ϕ is concave, in which case the effect is negative.*

Proof. Recall that

$$p = \frac{1}{a\mathbb{V}[\tilde{h}_{t+1}]\bar{X}} \left(1 + \phi(\bar{X}) - (1 + \mu)\mathbb{E}[(1 - \tilde{h}_{t+1})] \right).$$

A glance at the cryptocurrency equilibrium price given by (31) yields part (a) since from [Propositions 2](#) and [3](#), the numerator of p in (31) is non-positive. The sign of the variation of p with respect to \bar{X} is given by the sign of

$$-\bar{X}\phi'(\bar{X}) + \phi(\bar{X}) - \mu + (1 + \mu)\mathbb{E}[\tilde{h}_{t+1}].$$

This sign is negative if $\phi(\cdot)$ is increasing and indeterminate otherwise. Yet, when $\phi(\cdot)$ is also concave, observe that since $\phi(0) = 0$ by assumption, it always holds that $-\phi(\bar{X}) + \phi'(\bar{X})\bar{X} \leq 0$. Moreover, since in equilibrium $\mathbb{E}[\tilde{h}_{t+1}] \leq \frac{\mu - \phi(\bar{X})}{1 + \mu}$, then $\mathbb{E}[\tilde{h}_{t+1}] < \frac{\mu}{1 + \mu}$. The result follows. \square

To understand part (a) of the Proposition, it is useful to refer to Eq. (21). When the function utility function V pertaining to old age is quadratic, namely $V(x) = \frac{1}{2}ax^2 + bx + c$, a glance at this equation allows us to see that $(1 - \tilde{h})^2$ is included on its right-hand side. But the expected value of this expression is the variance $\mathbb{V}(\tilde{h}_{t+1})$ of the hack risk. Moreover, the sign of the coefficient of the variance of \tilde{h}_{t+1} is negative (the coefficient is equal to a and by assumption, $a < 0$). Therefore, the right-hand side of Eq. (21) is *ceteris paribus* a decreasing function of the hack risk. Glancing once again at this equation, we see that when the hack risk increases, then the price of cryptocurrency must decrease for the equation to be satisfied (of course, this is a partial equilibrium effect, as the value of money z is also affected).

Regarding parts (b) and (c) of the Proposition two effects are at work. Suppose indeed that the marginal cost of holding cryptocurrency is constant (that is, ϕ is linear). Then the only effect at work is a standard quantity theory one. That is, as the stock of cryptocurrency increases, its price must decrease for individual to be willing to detain it. When the marginal cost of holding cryptocurrency is not constant, a second effect is at work. In particular, when a change in the stock of cryptocurrency increases the marginal cost of this currency, the young savers try to hold more central bank currency. Hence, to induce them to hold a growing stock of cryptocurrency it is then necessary for its price to further decrease.

It is noteworthy, however, that an increase in cryptocurrency can in principle lead to an *increase* in its price, as suggested by part c of the Proposition. Suppose that ϕ is a non-increasing convex function, which means that there are economies of scales in holding the cryptocurrency. Then an increase in the stock of cryptocurrency decreases its marginal cost for young agents. This will lead them to substitute this currency for the central bank one. When this effect is stronger than the traditional quantity theory one, an increase in the stock of cryptocurrency may result in an increase of its price. That this is possible is illustrated by the next example.

Example. Assume that $\phi(\bar{X}) = \max\{\bar{\phi} - \bar{X}, 0\}$ (the cost of holding cryptocurrency is positive whenever $\bar{X} < \bar{\phi}$ and is given by a non-increasing convex function). Then the sign of the variation of the

equilibrium cryptocurrency price is the same as the sign of

$$\phi'(\bar{X})\bar{X} - \phi(\bar{X}) + \mu - (1 + \mu)\mathbb{E}[\tilde{h}_{t+1}]. \quad (35)$$

Assuming that $\phi(\bar{X}) > 0$, an increase in cryptocurrency leads to a rise in its prices when

$$\mu - (1 + \mu)\mathbb{E}[\tilde{h}_{t+1}] < \bar{\phi}. \quad (36)$$

We now consider the properties of the equilibrium value of central bank currency.

Proposition 5.

- (a) The equilibrium value of central bank currency increases with a mean-preserved increase in the risk of cryptocurrency.
- (b) When $\phi(\cdot)$ is increasing, the equilibrium value of central bank currency increases with the stock of cryptocurrency; it is a non-increasing function of this stock if so is $\phi(\cdot)$.

Proof. The Proposition follows directly from inspecting Eq. (32). \square

According to part (a) the prices of cryptocurrency and central bank currency change in opposite ways when the risk of theft increases. To understand this result one must take into account that savings must be invested either in one or the two currencies. The same remark explains part (b). What is interesting, however, is that while an increase in the stock cryptocurrency has an indetermined effect on its own price when $\phi(\cdot)$ is non-increasing, the effect on the price of the central bank currency is *always* determinate.

5. Interim dynamics

In the preceding section, we have focused on a steady-state equilibrium. We have indeed assumed that such an equilibrium is reached when cryptocurrency currency achieves its maximum values (that is, at date T). There are, of course, many other kinds of equilibria. But the steady-state one seems relevant, if only because the fundamentals of the model cease to change from date T on. What remains to be done is the study of the dynamics that occurs *before* date T . To do this, we have to proceed with backward induction, starting from the steady-state equilibrium at date T .

The next proposition gives in a special case a sequence of equilibrium prices from date 0, when cryptocurrency is introduced, to date T , when the stock of cryptocurrency ceases to grow. The values of cryptocurrency across time are obtained from the function $\rho : X_{t+1} - X_t = \rho(X_t)$, where $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is such that $X_T = \bar{X}$, X_0 being given.

Proposition 6. Assume that the utility function $U(\cdot)$ is linear and that $V'(\cdot)$ is positive. Then a sequence of equilibrium values $(p_t, \hat{z}_t)_{t=0}^\infty$ is given by the list of prices satisfying

$$p_t = \frac{1}{1 + \phi(X_t)} p_{t+1} \times \mathbb{E} \left[V' \left(e^0 + (1 - \tilde{h}_{t+1}) p_{t+1} X_t + \frac{\hat{z}_{t+1}}{1 + \mu} \right) (1 - \tilde{h}_{t+1}) \right] \quad (37)$$

$$\hat{z}_t = \frac{\hat{z}_{t+1}}{1 + \mu} \mathbb{E} \left[V' \left(e^0 + (1 - \tilde{h}_{t+1}) p_{t+1} X_t + \frac{\hat{z}_{t+1}}{1 + \mu} \right) \right]. \quad (38)$$

for all $t = 0, \dots, T-1$, and (p_t, \hat{p}_t) be the steady-state equilibrium system from date T onwards.

Proof. It is clear that the sequence gives equilibrium prices from date T on. Then consider date $T-1$. Since $V'(\cdot)$ is positive, Eqs. (37) and (38) yield positive prices. These two equations are nothing but the first-order conditions (14) and (15) for the choices of cryptocurrency and central bank currency of the young agent at date $T-1$. Since this agent's objective is concave, these necessary conditions ensure that holding the available stocks X_{T-1} and m_t is an optimal choice at date $T-1$ (because $\hat{z}_t = \hat{p}_t m_t$). By Walras law, since two out of three markets are balanced, then the good market is balanced as well. The same reasoning can be applied inductively for date $T-2, T-3, \dots, 0$.¹⁴ \square

6. Conclusion

We have studied the valorization of cryptocurrency in an overlapping generations model where individuals can also hold central bank currency. We have paid attention to three features of cryptocurrency: it is a risky asset (because of frauds and hacks), which one acquires by bearing transaction costs, and its supply reaches its upper limit in a finite time.

We have analyzed both the steady state and the dynamics that leads to it in a finite time. We find that the effect of an increase in cryptocurrency on the currency prices notably depends on the transaction costs borne by people to buy cryptocurrency. In particular, when these costs are given by a decreasing convex function (that is, there are economies of scale in holding cryptocurrency), an increase in cryptocurrency may result in an increase in its price.

There are two natural topics deserving future research. One topic is the study of the equilibria when there are several private cryptocurrencies, with different risks of hacking, and different upper limits. Another topic is the study of equilibria when the central bank issues its own cryptocurrency, with a stochastic upper-limit (reflecting, *i.e.*, random shifts in policy-makers' preferences).

Acknowledgments

This research has been conducted within the Research Chair "Digital Finance" under the aegis of the Risk Foundation, a joint initiative by Groupement des Cartes Bancaires CB, La Banque Postale, Telecom Paris and Paris II Panthéon-Assas University.

Appendix

Proof of Proposition 3. If (p, \hat{z}) is a stationary pair of values, then it satisfies the following conditions¹⁵

$$1 = \frac{1}{1 + \phi(\bar{X})} \times \mathbb{E} \left[\left(a \left(e^0 + (1 - \tilde{h}_{t+1}) p \bar{X} + \frac{\hat{z}}{1 + \mu} \right) + b \right) (1 - \tilde{h}_{t+1}) \right] \quad (39)$$

$$1 = \frac{1}{1 + \mu} \mathbb{E} \left[a \left(e^0 + (1 - \tilde{h}_{t+1}) p \bar{X} + \frac{\hat{z}}{1 + \mu} \right) + b \right] \quad (40)$$

¹⁴ Date 0 is not special, except that we assume that cryptocurrency and central bank currency are initially held by the old agent living at that date.

¹⁵ Besides the feasibility constraint $e^0 + (1 - \tilde{h}_{t+1}) p \bar{X} + \frac{\hat{z}}{1 + \mu} \in [0, -\frac{b}{a}]$. Clearly, if $p, \hat{z} \geq 0$ then $e^0 + (1 - \tilde{h}_{t+1}) p \bar{X} + \frac{\hat{z}}{1 + \mu} \geq 0$. Now, if $e^0 + p \bar{X} + \frac{\hat{z}}{1 + \mu} \leq -\frac{b}{a}$, that will ensure that $e^0 + (1 - \tilde{h}_{t+1}) p \bar{X} + \frac{\hat{z}}{1 + \mu} \in [0, -\frac{b}{a}]$.

which is equivalent to

$$1 - \frac{1}{1 + \phi(\bar{X})}(ae^o + b)\mathbb{E}[(1 - \tilde{h}_{t+1})] = \frac{a}{1 + \phi(\bar{X})}\bar{X} \times \mathbb{E}[(1 - \tilde{h}_{t+1})^2]p + \frac{a}{1 + \phi(\bar{X})}\mathbb{E}[(1 - \tilde{h}_{t+1})] \times \frac{\hat{z}}{1 + \mu} \quad (41)$$

$$1 - \frac{1}{1 + \mu}(ae^o + b) = \frac{a}{1 + \mu}\bar{X}\mathbb{E}[(1 - \tilde{h}_{t+1})]p + \frac{az}{(1 + \mu)^2} \quad (42)$$

The linear system above can be written

$$aA_{\phi(\bar{X}), \mu} \begin{pmatrix} \bar{X} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ \hat{z} \end{pmatrix} = B \quad (43)$$

where the matrices $A_{\phi(\bar{X}), \mu}$ and B are defined by $A_{\phi(\bar{X}), \mu} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ with

$$\begin{cases} a_{11} = \frac{1}{1 + \phi(\bar{X})}\mathbb{E}[(1 - \tilde{h}_{t+1})^2] \\ \quad = \frac{1}{1 + \phi(\bar{X})}(\mathbb{E}[(1 - \tilde{h}_{t+1})]^2 + \mathbb{V}[(1 - \tilde{h}_{t+1})]) \\ a_{12} = \frac{1}{1 + \phi(\bar{X})}\mathbb{E}\frac{1}{1 + \mu}[(1 - \tilde{h}_{t+1})] \\ a_{21} = \frac{1}{1 + \mu}\mathbb{E}[(1 - \tilde{h}_{t+1})] \\ a_{22} = \frac{1}{(1 + \mu)^2} \end{cases} \quad (44)$$

and

$$\begin{cases} b_1 = 1 - \frac{1}{1 + \phi(\bar{X})}(ae^o + b)\mathbb{E}[(1 - \tilde{h}_{t+1})] \\ b_2 = 1 - \frac{1}{1 + \mu}(ae^o + b). \end{cases} \quad (45)$$

Recall that $\bar{X} \neq 0$. The matrix $A_{\phi(\bar{X}), \mu} \begin{pmatrix} \bar{X} & 0 \\ 0 & 1 \end{pmatrix}$ is invertible if and only if $A_{\phi(\bar{X}), \mu}$ is, i.e., if

$$\frac{1}{1 + \phi(\bar{X})}(\mathbb{E}[(1 - \tilde{h}_{t+1})]^2 + \mathbb{V}[(1 - \tilde{h}_{t+1})]) - \frac{1}{1 + \phi(\bar{X})}\mathbb{E}[(1 - \tilde{h}_{t+1})]\mathbb{E}[(1 - \tilde{h}_{t+1})] \neq 0 \quad (46)$$

i.e. if

$$\frac{1}{1 + \phi(\bar{X})}\mathbb{V}[\tilde{h}_{t+1}] \neq 0. \quad (47)$$

Moreover, the unique solution is given by

$$\begin{pmatrix} p \\ \hat{z} \end{pmatrix} = \frac{1}{a} \begin{pmatrix} \frac{1}{\bar{X}} & 0 \\ 0 & 1 \end{pmatrix} A_{\phi(\bar{X}), \mu}^{-1} B \quad (48)$$

i.e.

$$\begin{pmatrix} p \\ \hat{z} \end{pmatrix} = \frac{(1 + \phi(\bar{X}))(1 + \mu)^2}{a\mathbb{V}[\tilde{h}_{t+1}]} \begin{pmatrix} \frac{1}{\bar{X}} & 0 \\ 0 & 1 \end{pmatrix} M \quad (49)$$

with

$$M = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad (50)$$

or

$$M = \begin{pmatrix} \frac{1}{(1 + \mu)^2} & -\frac{1}{1 + \phi(\bar{X})}\mathbb{E}[(1 - \tilde{h}_{t+1})]\frac{1}{1 + \mu} \\ -\frac{\mathbb{E}[(1 - \tilde{h}_{t+1})]}{1 + \mu} & \frac{1}{1 + \phi(\bar{X})}\mathbb{E}[(1 - \tilde{h}_{t+1})^2] \end{pmatrix} \times \begin{pmatrix} 1 - \frac{1}{1 + \phi(\bar{X})}(ae^o + b)\mathbb{E}[(1 - \tilde{h}_{t+1})] \\ 1 - \frac{1}{1 + \mu}(ae^o + b) \end{pmatrix}. \quad (51)$$

The Proposition follows since in Eq. (49) a is negative. \square

References

- Athey, I., Sarukkai, V., Xia, J., 2016. Bitcoin Pricing, Adoption, and Usage: Theory and Evidence. Working paper.
- Biais, B., Bisière, C., Bouvard, M., Casamatta, C., Menkveld, A., 2018. Equilibrium Bitcoin Pricing. Working paper, TSE.
- Budish, E., 2018. The Economic Limits of Bitcoin and the Blockchain, WP. University of Chicago Booth School of Business.
- Chiu, J., Koepl, T., 2017. The Economics of Cryptocurrencies—Bitcoin and beyond. Working paper, Bank of Canada.
- Fernández-Villaverde, J., Sanches, D., 2019. Can currency competition work? J. Monetary Econ. 106, 1–15.
- Garratt, R., Wallace, N., 2018. Bitcoin 1, bitcoin 2.: An experiment with privately issued outside monies. Econ. Inq. 56 (3), 1887–1897.
- Hu, A.S., Parlour, C.A., Rajan, H., 2019. Cryptocurrencies: Stylized facts on a new investible instruments. Financial Manage. 48 (4), 1049–1068.
- Lagos, R., Wright, R., 2005. A unified framework for monetary theory and policy analysis. J. Polit. Econ. 113 (3), 463–484.
- Makarov, I., Schoar, A., 2020. Trading and arbitrage on cryptocurrency markets. J. Financ. Econ. 135 (2), 293–319.
- Saleh, F., 2018. Volatility and Welfare in a Crypto Economy. Working paper, McGill University.
- Schilling, L., Uhlig, H., 2019. Some simple bitcoin economics. J. Monetary Econ. 106, 16–26.
- Schmidt, K.D., 2003. On the Covariance of Monotone Functions of a RandOm Variable. Professoren Des Inst. Für Math. Stochastik, 2003.
- Schmidt, K.D., 2006. Versicherungsmathematik. Springer-Verlag.